Heat Transfer to Pseudoplastic Fluids in Laminar Flow

E. B. CHRISTIANSEN and S. E. CRAIG, JR.
University of Utah, Salt Lake City, Utah

A simple temperature dependent equation $\tau = K\,[\dot{S}\exp(\Delta H\ddagger/RT)]^n$ is proposed to represent the rheological properties of pseudoplastic fluids. This equation is combined with the differential equations describing steady state laminar flow and heat conduction to a moving fluid when natural convection and thermal energy generation are negligible. The resulting equations are solved numerically for heating of both Newtonian and non-Newtonian fluids in laminar flow in tubes of circular cross section with constant wall temperature. These numerical solutions are presented as generalized plots of Nusselt number vs. Graetz number with n and $(\Delta H\ddagger/R)$ $(1/T_i - 1/T_w)$ as parameters.

Extensive experimental data were obtained for two pseudoplastic fluids, and, in the range where natural convection was unimportant, the mean deviation between the computed Nusselt numbers and measured Nusselt numbers was ± 7%.

The processing of non-Newtonian fluids is of continually increasing industrial importance. These fluids are characterized by a nonlinear rheogram or shearing stress-shearing rate relationship. Emulsions, slurries, and polymeric melts, solutions, and dispersions are often importantly non-Newtonian. Of the many types of non-Newtonian fluids considered in the literature (see for example references *1, 23, 27*) pseudoplastic fluids are most commonly encountered and are the principal concern of this study. The rheograms for most of these pseudoplastic fluids are quite accurately represented by the equation suggested by Eyring et al. (*22*):

$$B\tau = \gamma\frac{\dot{S}}{A'} + \sinh^{-1}\frac{\dot{S}}{A'} \qquad (1)$$

The occurrence of the shear rate in both the linear and inverse hyperbolic sine terms of Equation (1) makes this relationship somewhat cumbersome for many engineering purposes. Consequently the empirical Ostwald-de-Waele (*23*) or power law equation

$$\tau = K'\dot{S}^n \qquad (2)$$

has often been used as an approximate representation of pseudoplastic rheology (reference *3* for example). In some cases Equation (2) has been found to fit rheological data as well as or better than Equation (1) (*19*).

Since most non-Newtonian fluids have relatively high viscosities, these and highly viscous Newtonian fluids are often processed while in laminar flow. A study of the literature indicated that methods for predicting heat transfer to pseudoplastic fluids in laminar flow did not adequately incorporate the effect of the viscosity-temperature dependency. The major purpose of this investigation was to determine a realistic viscosity-temperature dependency and, based on this, to develop accurate methods for the prediction of heat transfer coefficients for the heating of pseudoplastic fluids in laminar flow in tubes of circular cross section and with constant wall temperature.

S. E. Craig, Jr., is with Arizona State University, Tempe, Arizona.

LAMINAR-FLOW HEAT TRANSFER

If the fluid properties k and C_p are constant, if generation of thermal energy in the fluid is negligible, and if energy (heat) conduction and fluid flow are restricted to the radial and axial directions, respectively, then the equation which describes heat transfer to fluids flowing in steady state laminar motion in a tube is

$$\left(\frac{\partial T}{\partial L}\right)_r = \frac{k}{u\rho C_p}\,\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]_L \qquad (3)$$

Equation (3) may be solved if the velocity and the density are known or are assigned functions of the radius. By assuming ρ to be independent of r, such solutions have been obtained by Graetz (*5, 8, 9*) for a parabolic velocity distribution [Equation (4)] and for a uniform (plug flow) velocity distribution [Equation (5)]:

Graetz:

$$N_{Nu} = \frac{2wC_p}{\pi kL}\left(\frac{1 - 8\psi_1(X)}{1 + 8\psi_1(X)}\right)$$

Isothermal (C_p, ρ, μ, $k \neq f(T)$), Newtonian ($n = 1$) flow (4)

Graetz:

$$N_{Nu} = \frac{2wC_p}{\pi kL}\left(\frac{1 - \psi_2(Y)}{1 + \psi_2(Y)}\right)$$

Plug flow ($n = 0$) (5)

Similarly solutions of Equation (3) have also been obtained by Lyche and Bird (*14*) for isothermal, pseudoplastic, velocity distributions. These solutions are converging infinite series of exponential functions which are difficult to evaluate at large Graetz numbers.

Approximate asymptotic solutions to Equation (3) have been obtained by assuming the heat transfer to be controlled by a thin film next to the tube wall and the velocity distribution in this film to be linear. This technique was used by Lévêque (*13*) to obtain a solution for isothermal Newtonian flow [Equation (6)] and by Pigford (*21*) to obtain a solution for isothermal pseudoplastic flow [Equation (7)]. Vaughn (*15, 26*) obtained an asymptotic solution for plug flow [Equation (8)]:

Lévêque:

$$N_{Nu} = 1.75\,N_{Gz}^{1/3}$$

Isothermal Newtonian flow ($n = 1$) (6)

Pigford:

$$N_{Nu} = 1.75\left(\frac{3n+1}{4n}\right)^{1/3} N_{Gz}^{1/3}$$

Isothermal pseudoplastic flow (7)

Vaughn:

$$N_{Nu} = \frac{8}{\pi} + \frac{4}{\pi}N_{Gz}^{1/2}$$

Plug flow ($n = 0$) (8)

Early experiments indicated that neither Graetz's nor Lévêque's solutions were capable of correlating experimental data owing to failure to incorporate the effect of temperature on fluid properties. The variation of viscosity with temperature within the heated tube causes the velocity profile to be distorted from the assumed profiles. To accommodate the effect of the viscosity-temperature dependency Sieder and Tate (*25*) suggested the following empirical equation:

$$N_{Nu}\left(\frac{\mu_w}{\mu_b}\right)^{0.14} = 1.75\,N_{Gz}^{1/3} \qquad (9)$$

This equation satisfactorily correlates much of the Newtonian heat transfer data and becomes completely unrepre-

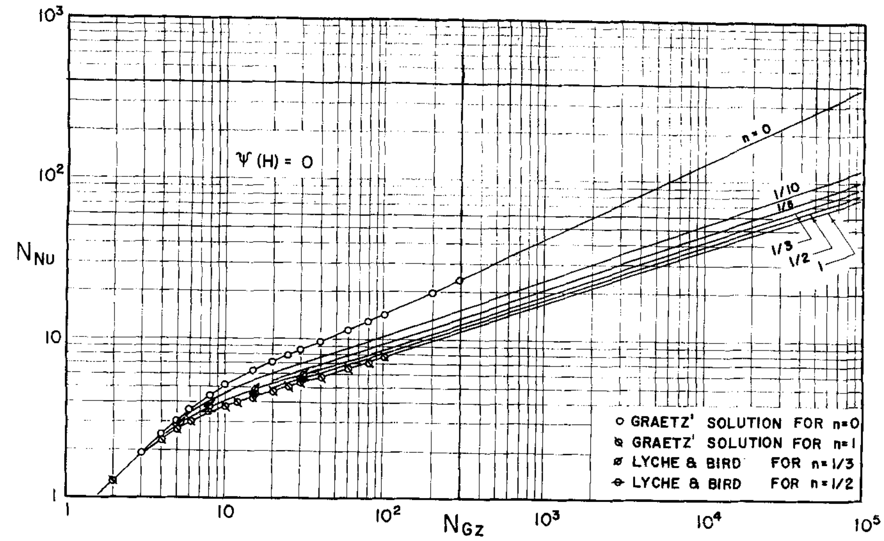

Fig. 1. Computed isothermal heat transfer data plots for laminar flow of Newtonian and pseudoplastic fluids in circular tubes compared with data from previous solutions for Equation (3).

sentative only at low values of the Graetz number and high values of $(T_w - T_i)$ where natural convection is important.

PSEUDOPLASTIC HEAT TRANSFER

Since Equation (7) erroneously predicts values for N_{Nu} approaching infinity as n becomes small (as the flow approaches plug flow), Vaughn (*15, 26*) substituted Δ for $(3n + 1)/4n$ in Equation (7). The factor $\Delta^{1/3}$ is defined as $[(3n + 1)/4n]^{1/3}$ for most values of n but approaches the ratio of the Nusselt number for plug flow [Equation (8)] to the Nusselt number for parabolic flow [Equation (6)] as n becomes small. Vaughn then added an empirical Sieder-Tate type of term to his equation to allow for velocity profile distortion due to variation of viscosity with temperature and obtained the equation

$$N_{Nu} = 1.75\, \Delta^{1/3}\, N_{Gz}{}^{1/3} \left(\frac{v_b}{v_w}\right)^{0.14} \tag{10}$$

In the limiting case of $n = 0$ plug flow occurs. Also as v_b/v_w (or μ_b/μ_w) approaches infinity, the flow becomes plug flow. It will be demonstrated later that in the case of laminar flow heat transfer and in the absence of natural convection effects (which may be relatively large at low N_{Gz}) the absolute upper limit of Nusselt numbers must be given by the plug flow case whether the velocity profile distortion is caused by non-Newtonian behavior or by variation of viscosity with temperature, or by both. Therefore both the Sieder-Tate equation and Vaughn's equation have a theoretical limitation in generality, since in each case Nusselt numbers are predicted which approach infinity as v_b/v_w (or μ_b/μ_w) approaches infinity.

MATHEMATICAL DEVELOPMENT

The advent of high-speed digital computers permits the solution of Equation (3) with more realistic velocity distributions. These distributions are obtained from a combination of force balance and a temperature dependent relationship representing the rheology of the fluid.

Various empirical and theoretical relations have been developed to correlate the effects of temperature upon the viscosities or consistencies of fluids. For Newtonian fluids the best known of these relations is the Arrhenius equation:

$$\tau = \mu' e^{\frac{\Delta H^\ddagger}{RT}} \dot{S} \tag{11}$$

Eyring (*10*) derived the equivalent of Equation (11) from the theory of absolute reaction rates, interpreted $\Delta H^\ddagger$ to be the activation energy per mole for flow, and showed that μ' is a very weak function of temperature.

Eyring and Ree (*7, 22*) suggest the following temperature dependent form of Equation (1) for pseudoplastic fluids:

$$B\tau = \gamma \frac{\dot{S}}{A} e^{\frac{\Delta H^\ddagger}{RT}} + \sinh^{-1} \frac{\dot{S}}{A} e^{\frac{\Delta H^\ddagger}{RT}} \tag{12}$$

By analogy to Equation (12) the Ostwald-deWaele power equation in temperature dependent form becomes

$$\tau = K\left(\dot{S}\, e^{\frac{\Delta H^\ddagger}{RT}}\right)^n \tag{13}$$

where K, n, and $\Delta H^\ddagger/R$ are empirical constants, presumably independent of temperature.

Although empirical Equation (13) fits much data adequately for most purposes (*4*). It is also in a convenient form for use in the subsequent mathematical analysis and was adopted to represent pseudoplastic rheology in this study.

If flow through a cylindrical tube is at a steady state, is axial only, and if there is no acceleration of the fluid passing through the elemental annulus, then a force balance yields

$$-\frac{dP}{dL} = \frac{1}{r}\frac{d(r\tau)}{dr} \tag{14}$$

Since P is not a function of r, and since, to a first approximation, τ is not a function of L (*11*), Equation (14) may be integrated between appropriate limits and combined with Equation (13) to yield

$$u = R_w \left[\frac{\tau_w}{K}\right]^{\frac{1}{n}} I_1 \tag{15}$$

The total flow in the pipe is given by

$$W = 2\pi\rho \int_0^{R_w} r u\, dr =$$

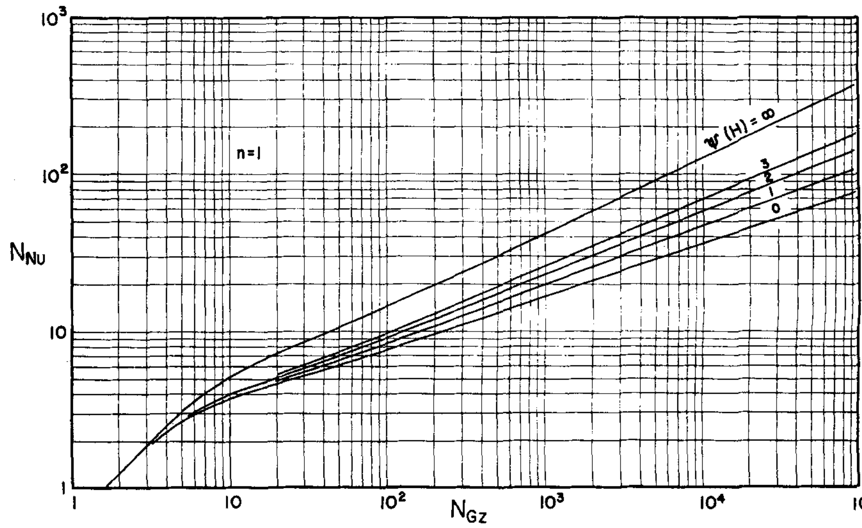

Fig. 2. Plots of computed data for heat transfer to Newtonian fluids in laminar flow.

$$2\pi \rho R_w^3 \left[\frac{\tau_w}{K}\right]^{\frac{1}{n}} I_2 \qquad (16)$$

The quantity $[\tau_w/K]^{\frac{1}{n}}$ is eliminated by combining Equations (15) and (16) to obtain the velocity distribution equation

$$u = \frac{W}{2\pi R_w^2 \rho} \frac{I_1}{I_2} \qquad (17)$$

This velocity distribution and the definitions of t, y, and N_{Gz} are substituted into Equation (3) to obtain the combined heat transfer and flow equation

$$\frac{\partial t}{\partial \left(\frac{1}{N_{Gz}}\right)} = 2\pi \frac{I_2}{I_1} \left(\frac{1}{y} \frac{\partial t}{\partial y} + \frac{\partial^2 t}{\partial y^2}\right) \qquad (18)$$

This equation, together with appropriate boundary conditions, describes the temperature distribution within the tube as a function of the Graetz number, $(T_w - T_i)/T_i$, n, and $\Delta H^{\ddagger}/RT_w$. The boundary conditions used in this work are a constant wall temperature and a constant inlet temperature. Implicit in the development of Equation (18) are the assumptions of well-developed isothermal laminar flow at the inlet and the temperature independence of C_p, k, $\Delta H^{\ddagger}/R$, ρ, K, and n.

Although analytic solutions to Equation (18) are unknown, numerical solutions were obtained through standard finite difference techniques. The general procedure was to divide the reduced radius into j equal parts of δy each and then to start at a reduced length $1/N_{Gz}$ equal to zero and to compute the temperature distribution for steps of length $\delta(1/N_{Gz})$. Second-order approximations were used to evaluate all partial derivatives in Equation (18). The result of these second-order approximations was a series of j simultaneous algebraic equations in $t(y, N_{Gz})$ to be solved for each step of $\delta(1/N_{Gz})$. This set of simultaneous equations was found to be especially adaptable to solution by the Crout reduction method (*12*). When the reduced temperature distribution had been found at each reduced length, the Nusselt number was determined from the solution of the equation

$$N_{Nu} = \frac{1}{\pi} N_{Gz} \frac{I_3}{1 - 0.5\, I_3} \qquad (19)$$

Equation (19) may be derived directly from the basic relations

$$q = w\, C_p\, (T_o - T_i) \qquad (20)$$

$$q = h_a \pi D\, L \left(T_w - \frac{T_i + T_o}{2}\right) \qquad (21)$$

$$q = 2\pi C_p \rho \int_0^{R_w} ru\, (T - T_i)\, dr \qquad (22)$$

and the definitions of y, t, N_{Nu}, and N_{Gz}.

Machine programs for a Datatron 205 digital computer were developed to solve the finite difference forms of Equations (18) and (19). By experimentation with these programs values of increments in y and $(1/N_{Gz})$ were determined which would yield essentially exact solutions of Equation (18) for Nusselt numbers up to a value of about 100 for the pseudoplastic cases and up to 1,000 for the plug flow case. With increasing values of N_{Gz} increasingly small increments in y next to the tube wall must be employed in order to maintain accuracy. While machine costs introduced a limit on the smallness of increments in y employed, the errors in the results at $N_{Gz} = 10^5$ are believed to be no greater than 1 or 2%.

From the mathematical analysis both $\psi(H) = (\Delta H^{\ddagger}/R)(1/T_i - 1/T_w)$ and $(T_w - T_i)/T_i$ were found to be independent parameters. In order to determine the effects of each of these parameters calculations were made for a Newtonian case having $\psi(H) = 1$, but with $(T_w - T_i)/T_i$ having the values of 0.1 and 0.2. The Nusselt numbers calculated with $(T_w - T_i)/T_i = 0.2$ were less than those for $(T_w - T_i)/T_i = 0.1$, but by no more than 0.4%. These results show that $\psi(H)$ is sufficient to characterize the effect of the temperature-viscosity dependence. Therefore on all subsequent computer runs only one value of $(T_w - T_i)/T_i$ was used for each value of $\psi(H)$.

THEORETICAL RESULTS

The theoretical results from the machine computations are shown graphically in Figures 1 through 5. The

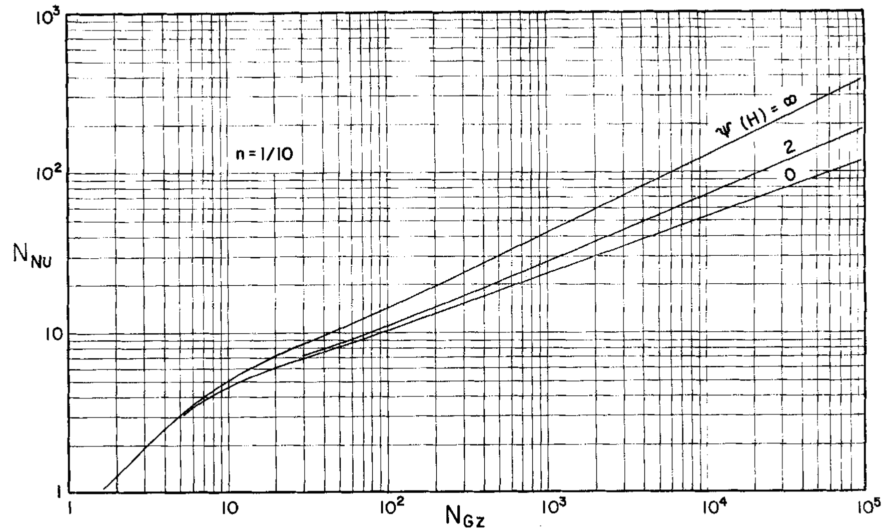

Fig. 4. Plots of computed data for heat transfer to pseudoplastic fluids in laminar flow; $n = 1/10$.

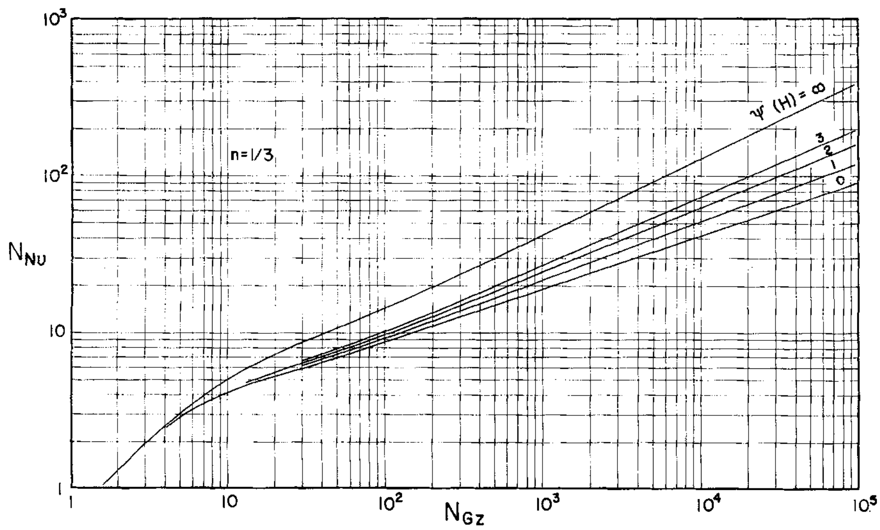

Fig. 3. Plots of computed data for heat transfer to pseudoplastic fluids in laminar flow; $n = 1/3$.

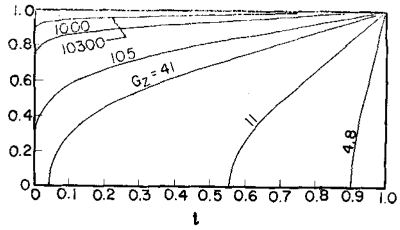

Fig. 5. Computed temperature distributions for laminar flow heating for $n = 1/3$ and $\psi(H) = 2$.

agreement between the computed data for parabolic flow [$n = 1$, $\psi(H) = 0$] and the values which Drew (5) gives for the Graetz parabolic flow solution [Equation (4)] is almost exact (see Figure 1). The corresponding asymptotic Lévêque solution [Equation (6)] is 5% lower at $N_{Gz} = 10$ and is approximately 2% lower at $N_{Gz} = 10^5$ than the computed solution. For the case of plug flow ($n = 0$) the agreement between the computed solution and Graetz's solution [Equation (5)] is exact and agreement with Vaughn's asymptotic solution [Equation (8)] is almost exact for Graetz numbers larger than 3,000. The computed isothermal pseudoplastic heat transfer data [$n < 1$, $\psi(H) = 0$] also agree almost exactly with the Graetz type of solution of Lyche and Bird (*14*). Pigford's approximate asymptotic isothermal solutions [Equation (7)] agree to within ± 5% of these computed solutions.

Plots of computed data for heat transfer to Newtonian fluids ($n = 1$) whose viscosities are temperature dependent according to Equation (13) [$\psi(H) \neq 0$] are shown in Figure 2. The factor $\psi(H)$ is analogous in purpose to the Sieder-Tate factor μ_b/μ_w, and these factors are related through Equation (11) as follows:

$$\frac{\mu_b}{\mu_w} = \exp\left[\frac{\Delta H^\ddagger}{R}\left(\frac{1}{T_b} - \frac{1}{T_w}\right)\right]$$

$$= \exp\left[\frac{\Delta H^\ddagger}{R}\left(\frac{1}{T_b} - \frac{1}{T_i}\right) + \psi(H)\right] \quad (23)$$

A comparison between the calculated curves and the Sieder-Tate equation [Equation (9)] shows that for the typical case of $\psi(H) = 2$, $n = 1$, the Sieder-Tate equation predicts values which are 12% high at $N_{Gz} = 50$, correct at $N_{Gz} = 500$, and 11% low at $N_{Gz} = 5{,}000$. As $\psi(H)$ is increased, the errors in the Sieder-Tate correlation become greater, and, conversely, as $\psi(H)$ is decreased, the errors become less.

In Figures 3 and 4 are shown typical calculated relationships between N_{Nu} and N_{Gz} as a function of the parameter $\psi(H)$ for heat transfer to pseudoplastic fluids ($n < 1$) whose viscosities are temperature dependent according to Equation (13). Vaughn's endeavor to allow for distortion of velocity profiles in consequence of the temperature dependence of viscosity by multiplying a modification of Pigford's equation with a Sieder-Tate type of term [Equation (10)] is not consistent with the results presented in Figures 2, 3, and 4. This may be demonstrated by comparing the ratios of the Nusselt numbers for a non-Newtonian case to the Nusselt numbers for a Newtonian case at the same values for the Graetz number and $\psi(H)$. For example when $\psi(H) = 0$, the ratio of the computer-calculated Nusselt number for $n = 1/10$ to that for $n = 1$ [which ratio by Equation (10) is independent of N_{Gz}] is 1.42. This corresponding ratio for $\psi(H) = 2$ is only 1.23, while Equation (10) predicts that this ratio should have a value of 1.48 at both $\psi(H) = 0$ and $\psi(H) = 2$, that is should be independent of v_b/v_w and consequently $\psi(H)$.

Typical computed outlet reduced temperature profiles are shown in Figure 5. These profiles clearly indicate the large temperature gradients which exist in the fluid at large values of the Graetz number.

EXPERIMENTAL EQUIPMENT

In order to test the results of the theoretical study large scale fluid-flow and heat transfer equipment was designed and constructed. This equipment is shown schematically in Figure 6. Each of the test sections had five double-shell steam jackets so arranged that the heated length could be varied between approximately 1 and 20 ft. The baffle type of fluid mixer in the inlet of the 15-ft. calming section was designed to destroy the temperature gradients established in the cooler. The small motorized mixer was placed in the outlet of the test sections to improve fluid temperature measurements at the outlet. The instrumentation included calibrated precision-grade thermometers which were mounted in the inlets to the calming sections and downstream of the outlet mixers. Numerous copper-constantan thermocouples were soldered into grooves in the copper test sections. These grooves were 1 ½ in. long, 1/16 in. deep, and were completely filled with solder so that the outer contour of the pipe was not changed by the thermocouple installation.

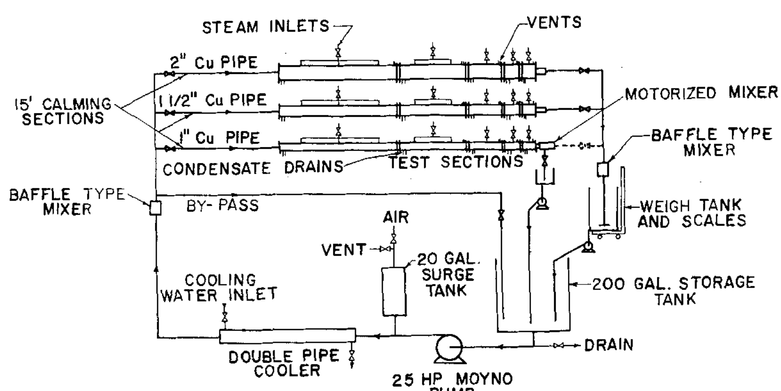

Fig. 6. Heat transfer apparatus.

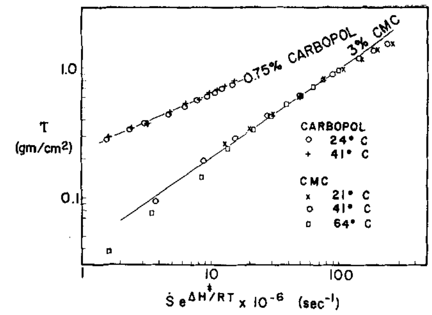

Fig. 7. Couette type of rotational viscometer data in temperature-dependent form for 0.75% CPM and 3% CMC in water.

Rheological data were obtained on a rotational viscometer which is a modification of the unit described by Salt (*24*). This viscometer consists of a rotating cup driven by a variable speed drive and a stationary bob suspended by a torsion wire. The cup and bob were mounted in a constant temperature oil bath. The accuracy of the viscometer was checked using a standard 60% sucrose solution, and the absolute viscosities based upon the physical measurements of the viscometer were within 1% of the values given in reference (*18*).

Complete heat transfer and rheological data were taken on a 3% water suspension of sodium carboxymethylcellulose (CMC) and a 0.75% water solution of carboxypolymethylene (CPM). These water suspensions, or solutions, were chosen because they form definitely non-Newtonian fluids which are fairly stable, because no health or fire hazard is associated with either, and because they are relatively easy to obtain and prepare in large quantities. In the concentrations used, the CMC was found to exhibit very slight visco-elastic effects and a very small yield stress. Also the consistency of the CMC suspension changed with aging, so that it was necessary to obtain rheological data at both the beginning and end of each set of heat transfer runs.

With the exception of the viscosity, the pertinent properties of the suspensions were taken to be the same as those of water. The heat capacity and density of the test

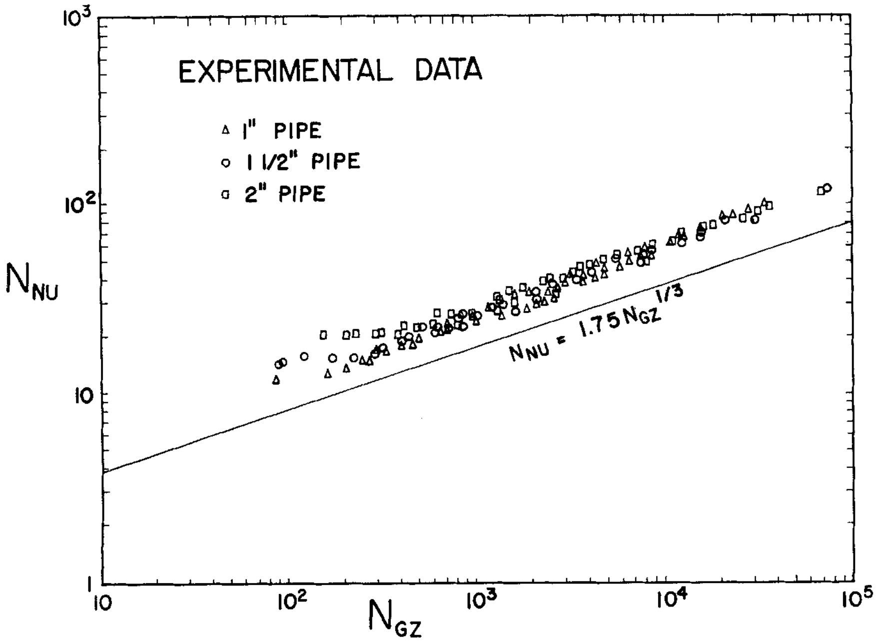

Fig. 8. Plots of experimental data for laminar flow heat transfer to 3% CMC in water.

fluids were determined by standard procedures and were found to be equal to those for pure water. Thermal conductivities for similar suspensions have been reported to be within 1 to 3% of the values for pure water (*2, 16, 20*).

HEAT TRANSFER PROCEDURE

After specified steady state conditions were obtained, as indicated by the constancy of all temperatures and pressures, the run was started. The condensate rate and flow rate were determined, and the inlet and outlet temperatures were read three times. At high flow rates it was impossible to obtain sufficient mixing of the heated fluid to make outlet temperature measurements meaningful. Therefore on these runs no outlet temperatures were taken, and no heat balances were obtained.

The precision of all measurements is shown in the following table. The main sources of error were found in the wall temperature measurements, the outlet temperature measurement, and the indeterminate errors connected with the condensate collection.

Measurement	Absolute precision	%
T_i	0.1°C.	
T_o	0.5°C.	
T_w	0.5°C.	
Sample weight		
slow flow	1.0 g.	±0.01
high flow	0.25 lb.	±0.3
Time of collecting sample	0.05 sec.	±0.5
Condensate weight	0.1 g.	±1.0
Time of collecting condensate	0.02 min.	±0.5

While the precision of the temperature measurement was good, the temperature rise through the heat transfer section was often very small, and hence a small error in the measurement of temperatures could cause a large error in the calculated value of $T_o - T_i$ and a correspondingly poor heat balance. For this reason the condensate rate was used in all cases in determining the amount of heat transferred.

For most runs the heat rate as determined from the inlet and outlet temperatures and the flow rate agreed with the heat rate as determined from the condensate rate to within ± 15%. Since the average uncertainty in $T_o - T_i$ was ± 10%, it is felt that the indeterminate error associated with the condensate rate was approximately ± 7%.

EXPERIMENTAL RESULTS

The experimental results are plotted in Figures 7 through 11. Typical viscometer data for 3% CMC and 0.75% CPM are presented in reduced form in Figure 7. The ability of the exponential term to account adequately for substantial viscosity changes with temperature is good for both of these fluids. The viscosity of the 3% CMC dispersion was approximately three times as great at 21°C. as at 64°C. Equation (13) accurately represents the range of CPM data. However in the application of Equation (13) to data such as the CMC data, the shearing rate range must be limited in accord with the accuracy required. Equation (12) would represent the CMC data more exactly. The following average rheological properties were determined:

Fluid	$\frac{\Delta H\ddagger}{R}$, (°K.)	n
3% CMC	3,840	0.67
0.75 CPM	3,540	0.45

Figure 8 shows the heat transfer data for 3% CMC. Considerable scatter and some stratification of data are noted in this plot. The heat transfer coefficient contained in the Nusselt number was calculated from the Equation (21) with q based upon the steam condensate collection. The value of the thermal conductivity used in the Nusselt number and the Graetz number was determined at the film temperature which is defined as $T_w/2 + T_i/4 + T_o/4$.

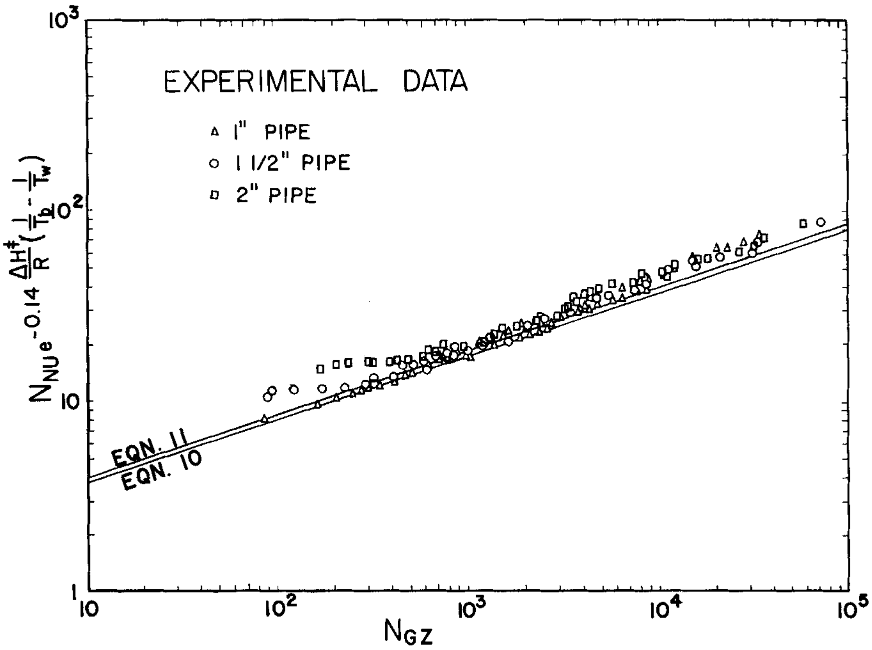

Fig. 9. A Sieder-Tate type of plot of experimental heat transfer data for 3% CMC in water.

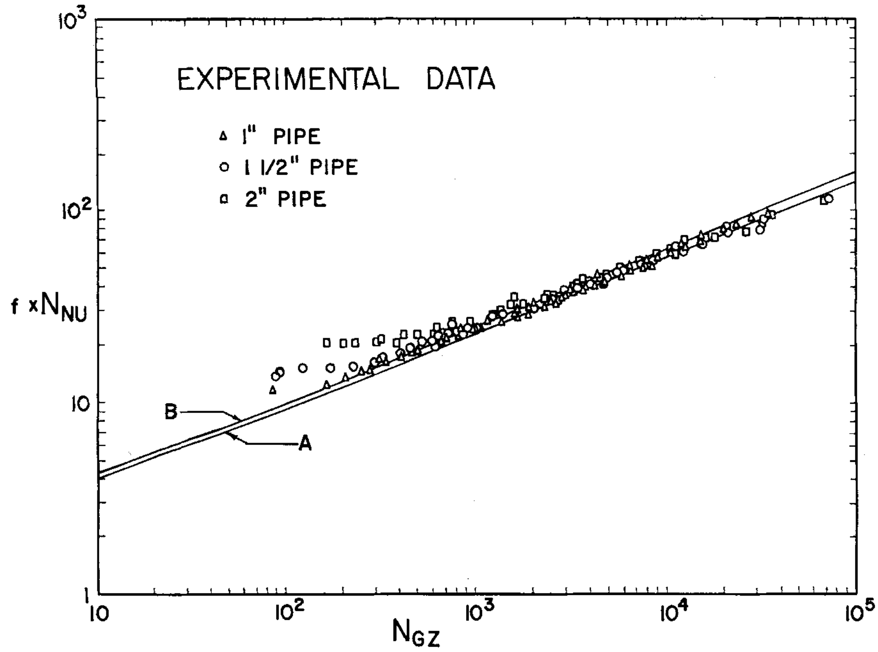

Fig. 10. Experimental heat transfer data for 3% CMC in water reduced to $\psi(H) = 2$ by the interpolation factor f. The computed data plots for $n = 1$, $\psi(H) = 2$ (curve A), and for $n = 1/3$, $\psi(H) = 2$ (curve B) are included for comparison.

In Figure 9 the CMC data are plotted with the ordinate changed to yield a Sieder-Tate type of plot. To avoid the confusion which results from the use of the factor $(\mu_b/\mu_w)^{0.14}$ for non-Newtonian fluids, the expression $\exp [0.14 (\Delta H^{\ddagger}/R) (1/T_b - 1/T_w)]$ is used. These two expressions are identical for the Newtonian case as shown by Equation (23). For non-Newtonian fluids the latter factor is unambiguous and is approximately the same as the ratio of apparent viscosities or consistencies to the 0.14 power. An exact representation of Vaughn's equation [Equation (10)] could not be drawn on this plot (Figure 9) because of the manner in which v_w/v_b is defined and because of slight changes in the value of n at different flow rates. However the line labeled Equation (10), which represents data calculated for the average value of n and with the assumption that $(v_w/v_b)^{0.14}$ is equal to the exponential term in the ordinate, is a fair representation of Vaughn's equation.

From a comparison of Figures 8 and 9 it is apparent that the Sieder-Tate correction reduces the scatter and stratification of the data, but neither the Sieder-Tate equation for Newtonian fluids nor Vaughn's equation adequately represents the data.

At low Graetz numbers the trend of data toward high values for the Nusselt number is caused by convective forces which become important in this range. The magnitude of this variation is substantially in agreement with that predicted by previous investigators (*6, 17*).

The experimental heat transfer data reduced to a common value of $\psi(H) = 2$ by means of a linear interpolation factor are plotted in Figures 10 and 11. For the case of $\psi(H) > 2$, f is defined by the equation

$$f = 1 + [\psi(H) - 2] \left[\frac{N_{Nu}(2)}{N_{Nu}(3)} - 1 \right] \tag{24}$$

where $N_{Nu}(2)$ is the computed Nusselt number for $\psi(H) = 2$ at the Graetz number and value of n for experimental data. Similarly, for the cases where $\psi(H) < 2$, f is defined by the equation

$$f = 1 + [2 - \psi(H)] \left[\frac{N_{Nu}(2)}{N_{Nu}(1)} - 1 \right] \tag{25}$$

Plots of computed data for $n = 1$ (curve A) and $n = 1/3$ (curve B) at $\psi(H) = 2$, are included in Figures 10 and 11 for comparison.

The experimental data agree exceptionally well with the curves representing calculated data except for those cases at low Graetz numbers, where natural convection is important, and for a few points at very high Graetz numbers. The data points which are obviously low at high Graetz numbers were all obtained in the 1 ft. long test sections. It is probable that the deviations here are due to difficulties in obtaining accurate wall temperatures and condensate rates in these very short lengths of pipe. Acceptable heat balances were not obtained for these very high Graetz number runs.

CONCLUSIONS

Digital computer solutions to the general energy flow equation have been presented which are applicable to heat transfer to Newtonian and non-Newtonian fluids in laminar flow in a tube under the following conditions:

1. The rheology conforms to the temperature reduced Ostwald-deWaele power law, with n, K, and $\Delta H^{\ddagger}$ independent of temperature.

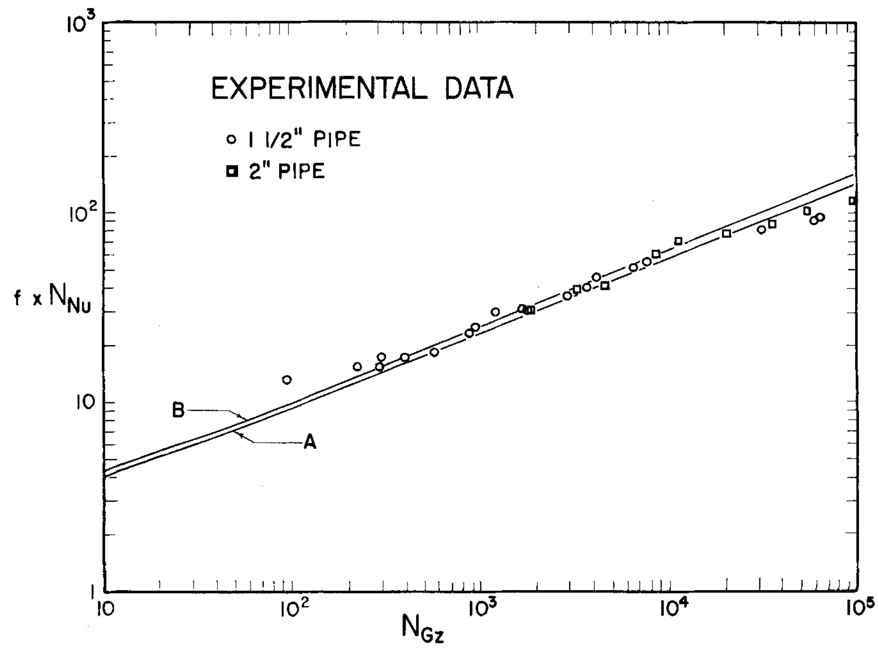

Fig. 11. Experimental heat transfer data for 0.75% CPM in water reduced to $\psi(H) = 2$ by the interpolation factor f. The computed data plots for $n = 1$, $\psi(H) = 2$ (curve A), and for $n = 1/3$, $\psi(H) = 2$ (curve B) are included for comparison.

2. Steady state conditions exist with a fully developed flow profile prior to entrance of heat transfer section.

3. Tube wall temperatures are constant.

4. ρ, C_p, and k are constant.

5. Fluid flow is longitudinal only.

6. Energy conduction is radial only.

7. Generation of thermal energy within the fluid is negligible.

The solutions have been demonstrated to be in excellent agreement with previous solutions for the corresponding conditions and in good agreement with experimental heat transfer data for the heating of a 3% aqueous suspension of CMC and a 0.75% aqueous suspension of CPM in laminar flow in tubes with L/D ratios varying from 6 to 230, fluid temperature increases up *to* 30°C., and temperature potentials up to 70°C. The solutions are believed to be essentially exact for plug flow for $N_{Nu} < 1{,}000$ and for Newtonian and pseudoplastic flow at $N_{Nu} < 100$.

These solutions can be confidently used in many heat transfer predictions for Newtonian and pseudoplastic fluids. The predictions will be conservative at $N_{Gz} < 1{,}000$ if natural convection is present.

Extensive, well-defined, and consistent experimental heat transfer data for two typical pseudoplastic fluids have been made available.

ACKNOWLEDGMENT

This report is based on a University of Utah, Department of Chemical Engineering, Graduate Thesis, "The Non-Isothermal Flow of Non-Newtonian Fluids," by S. E. Craig, Jr., June, 1959.

The authors wish to express their sincere appreciation to the National Science Foundation and to Union Carbide Nuclear Company for sponsoring this work. They also wish to thank the University of Utah for making available the Burroughs Datatron 205 digital computer used in the theoretical study.

NOTATION

(Units are given in metric system, but any consistent system can be used.)

A = constant in Equation (12), sec.$^{-1}$
A' = constant in Equation (1), sec.$^{-1}$
B = constant in Equations (1) and (12), sq. cm./dyne
C_p = heat capacity, cal./g. °K.
D = pipe diameter, cm.
e = base of natural logarithms, dimensionless
f = interpolation factor defined by Equations (24) and (25)
h_a = heat transfer coefficient based on average temperature difference
I_1 = $\int_y^1 y^{1/n} e^{-\Delta H^\ddagger/RT} dy$, dimensionless
I_2 = $\int_0^1 I_1 y dy$, dimensionless
I_3 = $\int_0^1 y\, t\, (I_1/I_2)\, dy$, dimensionless
K = constant in Equation (13), dyne (sec.)^n/sq. cm.
K' = constant in Equation (2), dyne (sec.)^n/sq. cm.
k = thermal conductivity, cal./cm. sec. °K.
L = length of heated tube, cm.
N_{Gz} = Graetz number ($w\, C_p/k\, L$), dimensionless
N_{Nu} = Nusselt number ($h_a\, D/k$), dimensionless
n = constant in Equations (2) and (13), dimensionless
P = pressure, dynes/sq. cm.
q = rate of heat transfer, cal./sec.
R = Boltzman constant per mole
R_w = pipe radius, cm.
r = local radius, cm.
$\dot{S}$ = shear rate, du/dx (du/dr for tube flow), (sec.$^{-1}$)
T = absolute temperature, °K.
t = reduced temperature $(T - T_i)/(T_w - T_i)$, dimensionless
u = local velocity, cm./sec.
v = consistency of pseudoplastic fluids (defined in reference *26*)
w = mass flow rate, g./sec.
y = reduced radius (r/R_w), dimensionless

Greek Letters

γ = constant in Equations (1) and (12), dimensionless
δ = small increment
Δ = interpolation factor in Equation (10), (see references *15, 26*)
$\Delta H^\ddagger/R$ = constant in Equations (11), (12), and (13), °K. [R is the Boltzman constant per mole and $\Delta H^\ddagger$ is interpreted (22) to be the activation energy for flow]
$\psi(H)$ = $(\Delta H^\ddagger/R)(1/T_i - 1/T_w)$, dimensionless
τ = shear stress, dyne/sq. cm.
ρ = density, g.-mass/cc.
μ = viscosity, $d\tau/d\dot{S}$ ($\tau/\dot{S}$ when $n = 1$) (g.-mass/cc.)
μ' = constant in Equation (11), (g.-mass/cm. sec.)
π = 3.1416
$\psi_1(X)$ = convergent infinite series of exponential functions in $\pi kL/4w\, C_p$
$\psi_2(Y)$ = convergent infinite series of exponential functions in $\pi kL/w\, C_p$

Subscripts

a = average between inlet and outlet
b = bulk
i = inlet
o = outlet
w = wall

LITERATURE CITED

1. Alves, G. E., D. F. Boucher, and R. L. Pigford, *Chem. Eng. Progr.*, **48**, 385 (1952).
2. Boggs, J. H., W. L. Sibbitt, *Ind. Eng. Chem.*, 47, No. 2, p. 289 (1955).
3. Christiansen, E. B., N. W. Ryan, and W. E. Stevens, *A.I.Ch.E. Journal*, **1**, No. 4, p. 544 (1955).
4. Craig, S. E., Jr., Ph.D. thesis, University of Utah, Salt Lake City, Utah (1959).
5. Drew, T. B., *Trans. Am. Inst. Chem. Engrs.*, **26**, 26 (1931).
6. Eubank, O. C., and W. S. Proctor, M.S. thesis, Mass. Inst. Technol., Cambridge, Massachusetts (1951).
7. Eyring, H., and T. Ree, *Nat'l Acad. Sci.*, **4**, No. 3, p. 118 (1955).
8. Graetz, L., *Annalen der Physik*, **18**, 79 (1883).
9. *Ibid.*, **25**, 337 (1885).
10. Glasstone, S., K. J. Laidler, and H. Eyring, "The Theory of Rate Processes," McGraw-Hill, New York (1941).
11. Hanks, Richard W., Ph.D. thesis, University of Utah, Salt Lake City, Utah (1960).
12. Hildebrand, F. B., "Introduction to Numerical Analysis," McGraw-Hill, New York (1956).
13. Lévêque, J., *Ann Mines* (*12*), **13**, 201, 205, 381 (1928).
14. Lyche, B. C., and R. B. Bird, *Chem. Eng. Sci.*, **6**, 34 (1956).
15. Metzner, A. B., R. D. Vaughn, and G. L. Houghton, *A.I.Ch.E. Journal*, **3**, 92 (1957).
16. Mortenson, P. C., B.S.Ch.E. thesis University of Utah, Salt Lake City, Utah (1960).
17. Morton, B. R., *Quart. J. Mech. App. Math.*, **12**, 410 (1959).
18. Perry, J. H., ed., "Chemical Engineers' Handbook," 3 ed., McGraw-Hill, New York (1950).
19. Petersen, A. W., Jr., Ph.D. thesis, University of Utah, Salt Lake City, Utah (1960).
20. Peterson, J. M., and M. W. Beckstead, B.S. Ch.E. thesis, University of Utah, Salt Lake City, Utah (1961).
21. Pigford, R. L., *Chem. Eng. Progr. Symposium Ser. No. 17*, **51**, 79 (1955).
22. Ree, T., and H. Eyring, *J. App. Phys.*, **26**, 793-809 (1955).
23. Reiner, M., "Deformation Strain and Flow," Interscience, New York (1960).
24. Salt, D. L., N. W. Ryan, and E. B. Christiansen, *J. Col. Sci.*, **6**, No. 2, p. 146 (1951).
25. Sieder, E. N., and G. E. Tate, *Ind. Eng. Chem.*, **28**, 1429 (1936).
26. Vaughn, R. D., Ph.D. thesis, University of Delaware, Newark, Delaware (1956).
27. Wilkinson, W. L., "Non-Newtonian Fluids," Pergamon, New York (1960).

Manuscript received April 1, 1960; revision received August 14, 1961; paper accepted August 16, 1961. Paper presented at A.I.Ch.E. Mexico City meeting.